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Modelling money market interest rates

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Introduction

Every Thursday the Court of the Bank of England meets and is confronted with a set of charts of financial developments in the previous week. One of these charts relates to the money market yield curve. It shows overnight, 1 week, 1 month, 3 month, 6 month and 12 month interest rates on the previous day (Wednesday), and one week earlier – a recent example is shown below (see Fig. 1). The frequency with which the two curves cross seemed sufficient to warrant collecting data for the 450 weeks between January 1978 and October 1986 as a preliminary to analysing the phenomenon econometrically.

There was a change in the operation of monetary policy in August 1981 and a break in the method of data collection in July 1982. Neither of these appears to have affected the frequency of such intersections or the distribution of what might be called pivotal maturities.

Of the total sample of 450 weekly observations, only the last 228 were used for formal econometric estimation. The full sample used for empirical work was split in this way because the raw data differed qualitatively in the two sub-samples; the first recorded the highest and lowest observations for the day of observation; the second recorded bid-ask spreads at a specific time during the day. For exploratory empirical work, the daily pairs were averaged, but only the bid-ask average data was thought to be sufficiently accurate to support econometric investigation.

In the full sample, pivoting occurred on 209, or 46 per cent, of occasions. Of these, 102 (49 per cent) were at maturities between 1 and 3 months, 73 (35 per cent) below 1 month and 34 (16 per cent) were beyond 3 months.

In this paper we discuss three issues:

(i) the explanation of the phenomenon – which is not very difficult
(ii) its possible econometric significance – which is not very great
but particularly
(iii) the modelling of the phenomenon econometrically – which proved problematic.

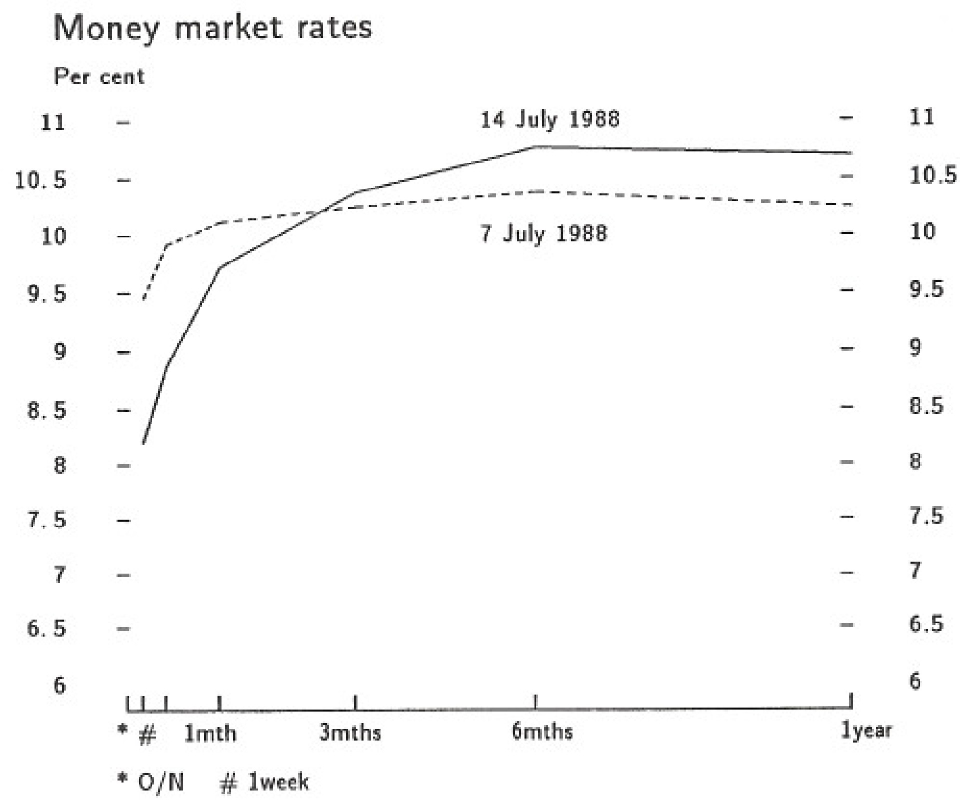

Figure 1

1. Explanation

The obvious explanation is that at any one time the Bank effectively pegs the rate at some maturity, say the 3 month rate, so that it is immune to the impact of 'news'. If the 'news' suggests to the market that rates will rise they will indeed do so at once beyond the pegged maturity if there is any expectation that the pegged rate will be raised in due course. Such an expectation would imply that capital losses would be anticipated on nearer maturities. This makes the rate on shorter term investments fall. Thus the pivot is about the pegged maturity (see Flemming (1989).[1]

On this basis all news generates pivoting, but this is overlaid by the fulfilment of expectations generated by the cumulative effect of previous news which can lead to changes over discrete intervals not involving intersections. One might expect these to be the smaller changes so that the negative 'news' effect would dominate the correlation of changes in long (26 week) and short (1 week) rates—but this

[1] The Bank's operating policies are in fact rather more complex than suggested by this initial hypothesis (see §3).

is not in fact true: the simple correlation is positive. At +0.2 it is, however, lower than that of changes in the weekly and the 3 month rate (+0.5) or in the 3 month and 6 month rate (+0.7).

2. Economic significance

Would such behaviour by the authorities be sensible? This is in principle questionable. Consider the extreme case in which the price of a very long bond were pegged in this way. Then when the authorities were expected to tighten policy all short rates would fall. This would not only imply an unnecessarily erratic path for short rates but would, if the authorities' response were slow enough, also lead to perverse movements in, e.g. monetary aggregates.

There is indeed a general point here about the consequences of the authorities allowing their responses to be so clear as to become (unconditionally) predictable. If the response relates to a quantity, harm is unlikely to be done but is liable to occur if the response relates to a price (e.g. temporary investment incentives might for this reason be inferior to public expenditure as an instrument for fine tuning aggregate demand—were that thought desirable). (See Flemming 1988.)

3. Formulating the model

Although we use discrete weekly data it is easier to formulate the model in continuous time. There are initially three key assumptions—although they might be relaxed without destroying the qualitative structure of the hypothesis.

(i) There is a maturity i the interest rate for which is fixed by the authorities in such a way that it never jumps. Moreover, $w < i < m$ where w is 1 week and m is 6 months.

(ii) The authorities have in mind a target interest rate $r(t)^*$ for this maturity and adjust $r_i(t)$ towards $r^*(t)$ at the fixed rate λ.

$$\dot{r}_i(t) = \lambda(r^*(t) - r_i(t)), \qquad (0 < \lambda). \tag{1}$$

(iii) The authorities use all available information efficiently so that

$$E_t(\dot{r}^*(t)) = 0. \tag{2}$$

The standard expectations hypothesis implies that knowledge of $r_i(t)$ and $r^*(t)$ is then sufficient to determine $r_w(t)$ and $r_m(t)$—

specifically, suppose that r is the instantaneous interest rate, then

$$r_w = \frac{1}{w}\int_0^w E_t(r(t+s))ds \tag{3a}$$

$$r_i = \frac{1}{i}\int_0^i E_t(r(t+s))ds \tag{3b}$$

$$r_m = \frac{1}{m}\int_0^m E_t(r(t+s))ds. \tag{3c}$$

Now once the news is in, all of the rates, r, r_w, r_i, and r_m will be expected to converge on r^* at the same rate λ. Thus in particular

$$E_t(r(t+s)) = r^*(t) - (r^*(t) - r(t))e^{-s\lambda} \tag{4}$$

whence

$$r_w - r^* = \frac{r - r^*}{w\lambda}(1 - e^{-w\lambda}) \tag{5a}$$

$$r_i - r^* = \frac{r - r^*}{i\lambda}(1 - e^{-i\lambda}) \tag{5b}$$

and

$$r_m - r^* = \frac{r - r^*}{m\lambda}(1 - e^{-m\lambda}). \tag{5c}$$

Neither r nor r^* is observable but by substituting from (5b) into (5a) and (5c) each of $r_w(t)$ and $r_m(t)$ can be written as a linear combination of the pegged rate $r_i(t)$ and the target rate $r^*(t)$.

$$(r_w - r^*) = (r_i - r^*)\frac{i}{I}\frac{W}{w} \tag{6a}$$

$$(r_m - r^*) = (r_i - r^*)\frac{i}{I}\frac{M}{m} \tag{6b}$$

where $W = i - e^{-w\lambda}$ etc. or, eliminating r^*,

$$\begin{aligned} r_w &= \left[\frac{\frac{i}{w}\frac{W}{I} - \frac{i}{m}\frac{M}{I}}{1 - \frac{i}{m}\frac{M}{I}}\right] r_i + \left[\frac{1 - \frac{i}{w}\frac{W}{I}}{1 - \frac{i}{m}\frac{M}{I}}\right] r_m \\ &= \left[\frac{\frac{W}{w} - \frac{M}{m}}{\frac{I}{i} - \frac{M}{m}}\right] r_i + \left[\frac{\frac{I}{i} - \frac{W}{w}}{\frac{I}{i} - \frac{M}{m}}\right] r_m \end{aligned}$$

which can also be written as

$$(I/i - M/m)(r_w - r_i) = (I/i - W/w)(r_m - r_i)$$

or

$$(r_w - r_i) = \frac{I/i - W/w}{I/i - M/m}\,(r_m - r_i) \tag{7a}$$

or

$$\omega = \gamma\mu, \tag{7b}$$

where $\omega = r_w - r_i$, $\mu = r_m - r_i$ and where

$$\gamma = \frac{I/i - W/w}{I/i - M/m}. \tag{8}$$

I, W, M are all of the form $X = 1 - e^{-x\lambda}$, which varies in both x and λ, which is as shown in Fig. 2.

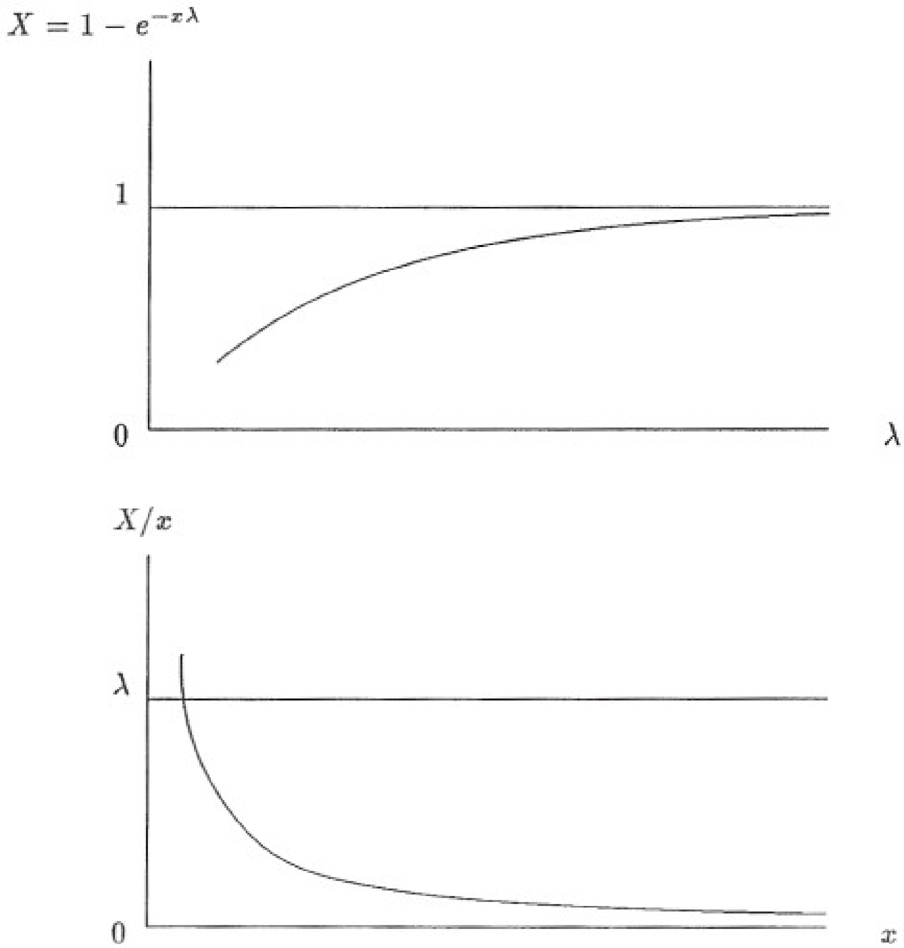

Figure 2

Thus if

$$w < i < m$$

$$W < I < M$$

but

$$W/w > I/i > M/m \quad \text{since} \quad \frac{X}{x} = \frac{1 - e^{-x\lambda}}{x}.$$

Thus the numerator of (8) is negative while the demonimator is positive for $\lambda > 0$.

Thus

$$\gamma < -1 \quad \text{iff} \quad 2 < \frac{i}{I}\left[\frac{W}{w} + \frac{M}{m}\right] \tag{9}$$

As $\lambda \to 0$, $X/x \to \lambda$, so the RHS of (9) tends to 2.

As $\lambda \to \infty$, $X/x \to 1/x$, so the RHS of (9) tends to $(1/w + 1/m)$ which is greater than 2 if i is greater than the harmonic mean of w and m. With $w = 1$ and $m = 26$ weeks, the critical value of i is 52/27, or less than two weeks. Thus we can be reasonably confident that $\gamma < -1$.

Differencing (7b) gives

$$\Delta r_w = \gamma \Delta r_m + (1 - \gamma)\Delta r_i \tag{10}$$

which implies that a negative γ only ensures that r_w and r_m move in opposite directions over a discrete interval if $\Delta r_i / \Delta r_m$ is small enough relative to the absolute size of γ ($|\gamma|$). On the maintained hypothesis both $|\gamma|$ and $\Delta r_i / \Delta r_m$ depend on the speed of adjustment λ, if λ were large relative to the observation interval observed pivots would be relatively rare.

The dynamics of the pivotal maturity rate itself (r_i) can be modelled by the discrete time analogue of (1), namely

$$\Delta r_i(t) = \frac{\lambda}{2}((r^*(t) - r_i(t)) + (r^*(t-1) - r_i(t-1)))$$

which gives equal weight, in explaining the changes in r_i over one period, to the opening and closing discrepancies.

Eliminating r_i from (6a&b), r^* can be expressed in terms of r_w and r_m as

$$(mW - Mw)r^* = mWr_m + Mwr_w \tag{11}$$

so that, substituting into (10)

$$\Delta r_i = \frac{\lambda}{2}\left[\frac{mW(r_m + r_{m,-1}) - Mw(r_w + r_{w,-1})}{mW - Mw} - (r_i + r_{i,-1})\right]$$

whence

$$r_i = \frac{2-\lambda}{2+\lambda} r_{i,-1} + \frac{\lambda}{2+\lambda}\left[\frac{mW(r_m + r_{m,-1}) - Mw(r_w + r_{w,-1})}{mW - Mw}\right]. \tag{12}$$

But from (7a)

$$r_i = \frac{m(Wi - wI)r_m + w(Im - iM)r_w}{mW - wM}. \tag{13}$$

From (12) and (13) a little rearrangement gives

$$(\frac{W}{w} - \frac{I}{i})\Delta r_m + (\frac{I}{i} - \frac{M}{m})\Delta r_w = \frac{\lambda}{2}\frac{I}{i}((r_m + r_{m,-1}) - (r_w + r_{w,-1})). \tag{14}$$

Equation (14) is our key relationship, embodying the whole model. If it could be estimated successfully one could recover from it the two key unknowns, λ and particularly i. If the estimated i falls between w and m we have empirical support for a pivot within the money market maturities.

As we shall see, estimating (14) presents several difficulties; there is no natural basis for assuming either Δr_m or Δr_w to be independent. This is related to the absence of any error structure at this stage. Moreover, there is no good reason to assume either λ or i constant.

Before attempting to estimate a relationship such as (14)—and testing the constraints on the coefficients required by our theory—we note that it has one very clear implication which is not consistently borne out by the data. This is that the money market yield curve should always be monotonic, the 3 month rate should always lie between the 1 week and the 6 month rates. In fact this requirement is breached roughly 25 per cent of the time.

This inconsistency between our theory and the data, although far from total, will necessarily be liable to introduce features into data-coherent equations that are difficult to explain. What we need is a richer theory capable of encompassing non-monotonic yield curves. Unfortunately such a theory is not readily to hand.

There are two possible explanations of the deviant observations which are compatible with the expectation hypothesis, augmented as necessary by less than perfect substitutability across maturities.

The first is that the administered rate (not necessarily the 3 month rate) is itself expected to follow a non-monotonic path *within the 6 month horizon*. This cannot be pursued unless one aspires to a very precise modelling of expectations which are here characterized in very simple terms.

The second case would arise if, say, the market wanted a yield curve flat at one level while the authorities insisted on holding an intermediate rate at some lower (or higher) level. If this were known, and believed to be sustainable, and to be going to be sustained, for the necessary period, and if 1 week, 3 month and 6 month bills are less than perfect substitutes, then a non-monotonic relationship could hold. Again the modelling of these conditions is beyond us.

What we can do is describe the known behaviour of the Bank's interventions where they differ from the simple inference drawn above as initial hypothesis. The Bank's behaviour deviated from that suggested by the initial hypothesis in two ways: the Bank did not 'peg' a particular maturity but relieved shortages in the money market by inviting offers of bills in four bands (1–14; 15–33; 34–63 and 64–91 days). For each band the Bank had a 'stop rate' corresponding to the highest price it was prepared to pay for paper of that maturity. The Bank did not necessarily deal at this price but might be able to cover the shortage at a lower price.

The effect of this is that the fulcrum of the pivoting process is not likely to be a single point but two points, the short end of band 1 when rates are expected to fall and the long end of band 4 when rates are expected to rise. This might suggest a bimodal distribution of the pivotal maturity but the data summarized in the Introduction do not support this implication, possibly because an intervening pivot is in fact possible if either the yield curve or the set of stop rates is suitably nonlinear. Nevertheless, the operating procedures do imply that the pivotal maturity is not only unlikely to be constant but is quite likely to move quite rapidly from one end to the other of the range of stop rates.

A second point is that the plausibility of the partial adjustment mechanism suggested for the pegged rate is qualified by the fact that the Bank knows that changes in its dealing rates are liable to lead to shifts in clearing bank base rates. At times this means that its stop rates cannot change for fear of dislodging base rates while at other times they may be required to move discretely in order to bring about a desired shift.

Moreover, the Bank's stop rates, like base rates, are conventionally moved in 1/2 per cent point steps. This is compatible with a smooth path for the expected future rate since the rate expected for a week hence would be 1/4 per cent higher if there were a 50/50 chance of a 1/2 per cent rise. Uncertainty, however, about the actual level of these interpolated 'expected rates' must be greater than when the expectation relates to a rate that might actually rule.

To build these last features into the model would require not only the modelling of the relationship between money market and base rates—which would not be too difficult; but of the authorities' desired base rates as well—which would be a major exercise in its own right.

Thus we embark on the confrontation of the model underlying equation (14) with the data in the knowledge that it does not embody anything like a full description of the Bank's operating procedures and that its expectational assumptions carry implications which are not supported by the data. In fact there is a second implication which is also amenable to direct testing and which raises some related questions. This is that under the expectations hypothesis it is clear that the variability of the 1 week rate r_w should exceed that of the 3 month and 6 month rates r_i and r_m. Indeed one would normally expect $V(r_w) > V(r_i) > V(r_m)$, where $V(\cdot)$ is some appropriate measure of variability, on the basis that r_i and r_m are successively longer moving averages of r_w with increasing smoothing built on. Of course our thesis is that r_i is subject to administrative smoothing. Thus $V(r_i) < V(r_m)$ might be explicable. Table 1 sets out the relative variation of levels and first differences.

Table 1. Standard deviation of money market rates

(a)	Whole sample			
	r_w	2.7	Δr_w	0.6
	r_i	2.6	Δr_i	0.4
	r_m	2.3	Δr_m	0.5
(b)	Estimation sample			
	r_w	1.5	Δr_w	0.5
	r_i	1.3	Δr_i	0.4
	r_m	1.1	Δr_m	0.3

While the ratio of $V(r_w)$ to $V(r_i)$ is smaller than might have been expected, our priors as to the rankings are fulfilled everywhere, except that $V(\Delta r_i) < V(\Delta r_m)$ in the whole sample—which might be induced by administrative smoothing.

4. Estimation

Equation (14) can be rewritten as

$$r_w\left[\left(1+\frac{\lambda}{2}\right)\frac{I}{i}-\frac{M}{m}\right]=\left[\left(1-\frac{\lambda}{2}\right)\frac{I}{i}-\frac{M}{m}\right]r_{w,-1}$$
$$-\left[\frac{W}{w}-\left(1+\frac{\lambda}{2}\right)\frac{I}{i}\right]r_m+\left[\frac{W}{w}-\left(1-\frac{\lambda}{2}\right)\frac{I}{i}\right]r_{m,-1} \qquad (15)$$

more generally, or,

$$r_w=a_0+\sum_{j=0}^{n}b_j r_{m,-j}+\sum_{j=1}^{n}c_j r_{w,-j}, \qquad (16)$$

where (16) reduces to (15) if

$$\begin{aligned}
a_0 &= 0\\
b_0 &= \left[\left(1+\frac{\lambda}{2}\right)\frac{I}{i}\frac{W}{w}\right]\Big/\left[\left(1+\frac{\lambda}{2}\right)\frac{I}{i}\frac{M}{m}\right]\\
b_1 &= \left[\frac{W}{w}-\left(1-\frac{\lambda}{2}\right)\frac{I}{i}\right]\Big/\left[\left(1+\frac{\lambda}{2}\right)\frac{I}{i}-\frac{M}{m}\right]\\
b_2 \ldots &= 0 \qquad (17)\\
c_1 &= \left[\left(1-\frac{\lambda}{2}\right)\frac{I}{i}-\frac{M}{m}\right]\Big/\left[\left(1+\frac{\lambda}{2}\right)\frac{I}{i}\frac{M}{m}\right]\\
c_2 \ldots &= 0.
\end{aligned}$$

Ntice that with $\lambda > 0$ and $W/w > I/i > M/m > 0$, $b_1 > 0$; b_0 and c_1 are not so easily signed although we do also have $b_1 > b_0 > -b_1$; $1 > c_1 > -1$ and $b_0 + b_1 + c_1 = 1$.

Equation (14) could also be written with r_m as the explanatory variable and a similar set of constraints on the coefficients.

Estimation presented two related problems. If r_m is included amongst the explanatory variables for r_w (and r_w amongst those for r_m) we have a problem of simultaneity—which can in principle be tackled by an IV approach. At the same time, although there is really only one relationship, it could be estimated with either r_w or r_m as the dependent variable.

The choice of potential instruments is severely limited in models of this type. Where r_m is an explanatory variable, valid instruments should be correlated with r_m but not with the error term. Since the latter includes all factors that may affect r_w but that are not included in the equation it is unlikely that any *contemporaneous* variable will qualify. Thus lagged interest rates were chosen, three lags of each of the rates included in the model and three lags on the 3 month and 1 year (interbank) rate. Provided that the error term is serially independent, the estimates obtained will be consistent. They are likely to be rather inefficient, however, given that changes in interest rates tend to be large and, assuming efficient markets, are not strongly correlated with past data.

Furthermore, we should be able to invert the estimated equation to obtain the results that would have been produced by estimation of the alternative form. However this 'invertability' condition is not satisfied by IV estimation in finite samples so the condition was imposed.

The invertability 'problem' is associated with the fact that we are estimating the relationship between two variables neither of which has any stronger claim than the other to be the independent variable. One way of estimating the relationship without giving priority to minimizing either vertical or horizontal errors is to minimize the sum of squared deviations measured perpendicularly to the fitted line. In many cases this procedure would be rendered arbitrary by the freedom to choose units of measurement of the respective variables. In this case, however, they all have the same natural units so that this objection fails.

In the case of the simple equation $y = \alpha + \beta x + \varepsilon$ it can be shown that the sum of the squared perpendicular errors is given by $(\widehat{\varepsilon}'\widehat{\varepsilon})/(1+\widehat{\beta}^2)$ and this, rather than the usual criterion function $\widehat{\varepsilon}'\widehat{\varepsilon}$ is minimized in perpendicular regression (see, e.g. Malinvaud 1979, pp. 9, 35, 85).

An analogous procedure was followed here in which the IV minimand

$$(\widehat{\varepsilon} H(H'H)^{-1}H'\widehat{\varepsilon})$$

was replaced by

$$\frac{(\widehat{\varepsilon}' H(H'H)^{-1}H'\widehat{\varepsilon})}{1+\widehat{\beta}^2}$$

where H is a matrix of instruments.

The estimation process involved a search over imposed values of $\widehat{\beta}$. Consequently all of the diagnostic statistics in the final estimated equation are conditional on a fixed $\widehat{\beta}$. In particular a t statistic for $\widehat{\beta}$ itself could not be obtained. An alternative procedure in which the two forms of the equation were estimated subject to the cross equation constraint on β by 3SLS produced almost identical results along with a full set of diagnostics.

5. Results

It rapidly became apparent that lags beyond one were not significant even in weekly data. The results reported here therefore relate only to first order lags.

Equation (16) was estimated in four different ways: by 0LS and by 2SLS, directly and also perpendicularly, with the results shown in Table 2.

Table 2

	0LS	2SLS	0LSP	2SLSP
a_0 (constant)	−1.0	−1.0	−2.0	−1.7
	(3.4)*	(2.4)	(4.0)	(4.2)
$b_0(r_m)$	+0.5	+0.5	+2.2	+1.7
	(5.9)	(1.6)	(imposed)	(imposed)
$b_1(r_{m,-1})$	−0.1	−0.1	−1.7	−1.2
	(1.1)	(10.3)	(17.7)	(15.4)
$c_1(r_{w,-1})$	+0.7	+0.7	+0.7	+0.7
	(16.2)	(16.2)	(9.8)	(12.0)
R^2	0.92	0.92	0.81	0.86
$\widehat{b}_0 + \widehat{b}_1 + \widehat{c}_1$	1.1	1.1	1.2	1.2

**t* statistics in parentheses.

The first thing to notice is that direct 0LS and 2SLS produce virtually identical results—there is in fact some difference lost in rounding.

The second thing to notice is that $b_0 + b_1 + c_1$ is 'near' to unity—although the constraint is strongly rejected by the data ($\chi^2 = 18.1$ against a critical value 3.8). Thirdly, c_1 does indeed lie between plus one and minus one. That is the extent of the good news as b_1 is consistently, though not always significantly, negative and smaller than b_0, a major shortcoming).

We also estimated equation (14) itself to produce direct estimates of λ and i, both by 2SLS and also perpendicularly. The results were as shown in Table 3. In this form the estimated coefficients do at least have the right sign and the implied estimated value of λ, the speed of adjustment, of 0.22 per week in both cases is quite plausible, albeit perhaps a little low, and quite well determined in the 2SLS

Table 3

	2SLS	2SLSP
Constant	−0.1	−0.1
	(0.5)	(0.2)
$d_1\left[\frac{2}{\lambda}\left[\frac{W/w-\frac{I}{i}}{I/i}\right]\right](\Delta_{rm})$	3.8	7.0
	(5.6)	(–)
$d_1\left[\frac{2}{\lambda}\left[\frac{I/i-\frac{M}{m}}{I/i}\right]\right](\Delta_{rw})$	1.5	0.4
	(1.1)	(0.21)
$\widehat{\lambda}$ [implied]	0.2	[0.23]
	(3.8)	
$\widehat{\imath}$ [implied]	15.4	[24.6]
	(7.8)	
R^2	0.02	0.02
DW	1.6	1.99

case for which we have a 't' statistic. The estimates of i, the pivotal maturity, are slightly less satisfactory, at least in the perpendicular 2SLS case. 24 1/2 weeks is far above the directly observed central frequency, which is much closer to the 15 weeks suggested by the conventional 2SLS. Both regressions have very low R^2 which suggests that the model accounts for relatively little of what is going on. The fact that the key coefficients are, nonetheless, well-determined reflects the relatively large number of observations arising from the use of a weekly data set.

6. Conclusions

This attempt to provide formal quantitative empirical support for a plausible interpretation of an observed phenomenon has been a limited success. Some support is forthcoming in the form of fairly plausible and well-determined parameter estimates—but the equation explains virtually none of the variation of the dependent variable.

The incompleteness of our success may well be related to three features of the data which we knew from the start that our simple model could not account for. These are:

1 the occasional non-monotonicity of observed money market yield curves

2 the associated deviation of the relative yield variabilities from those suggested by the theory, and

3 the non-constancy of the observed pivotal maturities.

The first two of these problems require a significant advance in modelling participants' expectations of the path of administered rates. The last might be met by amending the model to allow the two parameters λ and i (the adjustment speed and the pivotal maturity) to follow some dynamic stochastic processes. A very preliminary experiment, however, with a first order auto-regressive process for each parameter was not very encouraging—possibly for the reasons given in §3.

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